The method of generalized potentials for the synthesis of the communication network in the deterministic and nondeterministic cases

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Abstract: A new efficient algorithm for linear separable problem of synthesis of communication network, titled "method of generalized potentials", has been developed and validated. It is close to the well-known method of potentials for solving the classical transportation problem. The proposed algorithm has been developed for the Gale problem on demand and supply. The finiteness of the algorithm and the inability of the looping situations have been proved.

Key-Words: supply and demand problem, distribution of resources, linear programming, duality theory, method of potentials, spanning subtree.

1 Introduction

The classical method of potentials for the standard transport problem (STP) is well known. In its essence, it is a modification of the simplex method of solving the linear programming problem with reference to the STP. It allows, moving from some feasible basic solution, to obtain the optimal solution for a finite number of iterations [1]. There is a modification of this method for transport problems with arc capacities limitations [2]. In this paper, we consider the generalization of the potential method for the problem of optimal linear synthesis of a communication network in the problem of supply and demand. Initially, this method is presented and justified for a deterministic problem. Further, its application to a problem with undefined factors is considered.

We consider the problem of network synthesis for the Gale model on supply and demand [3]. Let us consider a given oriented graph with nodes from the set $P = \{p_i : i = 1,\ldots,\eta\}$ and arcs j from the set G. We shall also consider some set of nodes A (subset of P), $A = \{p_i : i = 1,\ldots,n\}$, which we call nodes of production. Consider also a set of nodes C (subset of P), $C = \{p_i : i = \eta - m + 1,\ldots,\eta\}$, which we will call nodes of consumption. Let us the remaining nodes form a set $B = \{p_i : i = n+1,\ldots,\eta - m\}$, which we will call the intermediate nodes. For each node $p_i \in A$ corresponds to some nonnegative production capacity function $\varphi_i(x) \geq 0$, $x \in X$, where X – the set of feasible distributions of resources. Furthermore, non-negative functions $\varphi_j(x) \geq 0$, $j \in G$, are known, which defines the arcs capacities, depending also on the distribution of resources [2]. Thus, resources are allocated as between nodes of production, thereby determining the production capacity and between the arcs of the network, determining their capacities. For nodes of product consumption demands - d_j are known. Further for simplification of the records we write $i \in A, i \in B, i \in C$ instead of $p_i \in A, p_i \in B, p_i \in C$ (Fig. 1)

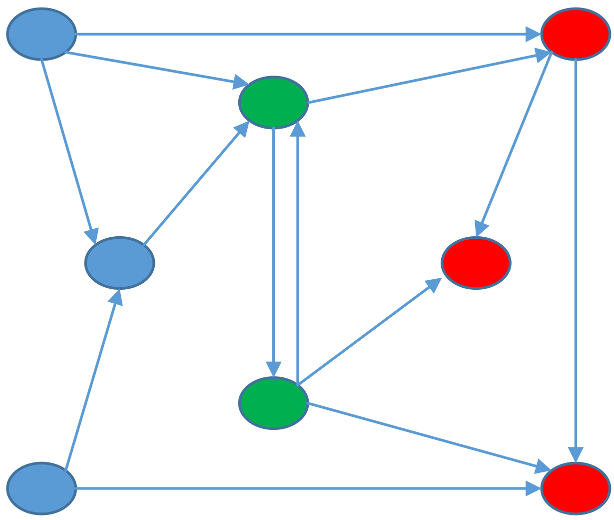

Fig. 1. The Gale model on supply and demand

It is known, that model with several nodes of production and limited stocks of product can be reduced to the problem of one node of production with an unlimited supply of product (Fig. 2). One new node is added - a production point with an unlimited supply of the product, which is linked by arcs to each of the production nodes. Functions of arc capacity of added arcs are functions of production capacities of the corresponding nodes.. The nodes of set A (nodes of production) become intermediate nodes. It is easy to show the equivalence of both problems.

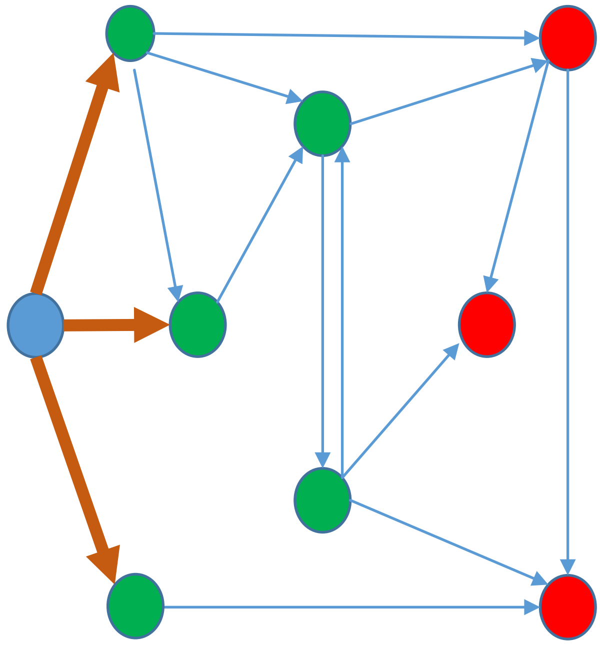

Fig. 2. Gale model modification.

The task is to consider possible distributions of resources and associated product flow, which due to product offerings in the nodes of production, satisfy the product demand in the nodes of consumption.

2 Problem Formulation

Mathematical formulation of the problem is the following:

$$\min_{x,y}\left(\sum_{j\in\Gamma} x_j\right), \qquad (1)$$

$$\sum_{j\in C(i)} y_j - \sum_{j\in D(i)} y_j = 0,\ i\in A\cup B,$$

$$\sum_{j\in D(i)} y_j - \sum_{j\in C(i)} y_j = d_i,\ \ i\in C,$$

$$y_j - a_j x_j \le b_j, \qquad j\in\Gamma,$$

$$x_j \ge 0,\quad y_j \ge 0, \qquad j\in\Gamma.$$

Where $a_j > 0$, $b_j \ge 0$, , $j\in\Gamma$.

Problem (1) is a linear programming problem. Here we are considering the dual problem;

$$\max_{\lambda,\mu}\left(\sum_{i\in C}\lambda_i d_i - \sum_{j\in\Gamma}\mu_j b_j\right) \qquad (2)$$

$$1-\mu_j a_j \ge 0, \qquad j\in\Gamma,$$

$$\lambda_{n_1(j)} - \lambda_{n_2(j)} + \mu_j a_j \ge 0, \qquad j\in\Gamma,$$

$$\mu_j \ge 0, \qquad j\in\Gamma.$$

For problems (1) and (2) the following theorem can be proved. It is based on applying well-known theorems of linear programming duality theory.

Theorem 1. For optimality of the vector *(x,y)* of problem (1) is necessary and sufficient the existence of a vector (λ,μ) satisfying the constraints of problem (2) and the associated vector *(x,y)* by the following expressions (the complementary slackness conditions):

$$1-\mu_j a_j = 0, when \qquad x_j > 0,\quad j\in\Gamma, \qquad (3)$$

$$\lambda_{n_1(j)} - \lambda_{n_2(j)} + \mu_j a_j = 0,$$

$$when \qquad y_j > 0,\quad j\in\Gamma,$$

$$\mu_j = 0,\ when \qquad y_j - a_j x_j < b_j \quad j\in\Gamma$$

The essence of the following algorithm consists of a sequential viewing of the extreme-point solutions (1), the values of functional for which monotonically decreasing. For a finite number of steps extreme-point solution of problem (1) is founded, for which there exists a vector (λ,μ) satisfying the constraints of problem (2) and ratios (3). According to the above theorem 1 the optimality of this solution follows.

3 Problem Solution

The description of the algorithm is presenting bellow. We introduce two definitions.

Definition 1. We call an arc *j* θ-arc, if $y_j = b_j > 0$, $x_j = 0$.

Definition 2. We call an arc *j* 0-arc, if $y_j = 0$.

The restrictions of the problem (1) can be reduced to an equivalent system of equalities specifying limits on the capacities of the arcs as $y_j - a_j x_j + z_j = b_j$ and adding the constraint

$z_j \geq 0, \quad j \in \Gamma$. Now we can make the following statement.

Statement 1. The vector *(x,y,z)* is a extreme-point solution of problem (1) if, and only if, when, after removal of all θ- arcs in the graph no cycle of arcs with non-zero flows remains.

Let us prove Statement 1. It is known, that the vector *(x,y,z)* is extreme-point solution if and only if the columns corresponding to the nonzero components form a linearly independent system of vectors. The matrix structure of the constraints in the problem (1) is shown in Fig. 3. Let us denote the matrix of incidence graph as *IN*.

The arc j is an θ-arc if and only if $z_j = 0, \quad x_j = 0$. Therefore, the columns corresponding to these variables are not included in the system. Then it is easy to see that in any combination of null vectors (representing zero vector) the column which corresponds to variable y_j, enters with a zero coefficient.

y	x	z
IN	0	0
1 0	$-a_1$ 0	1 0
1	.	1
1	.	1
1	.	1
0 1	0 $-a_n$	0 1

Fig. 3. The matrix structure of the constraints in the problem (1)

From the structure matrix of constraints, it is easy to see that the columns of the system are linearly independent if and only if when the columns of the incidence matrix of the system corresponding to non θ-arcs, are linearly independent. From graph theory, it is known, that this is equivalent to saying that these arcs form a forest. Which in its turn is equivalent to saying that these arcs do not form cycles. The proof is complete.

Definition 3. Arc j belonging to some spanning subtree, is called properly oriented if the beginning of the arc $n_1(j)$ belongs to the path connecting the end of the arc $n_2(j)$ with the root of the tree.

Let there be a spanning subtree with the set of arcs *GT*, which we shall call the current subtree. Let us also say that there is an admissible solution *(x,y)* of problem (1), such that:

1) arcs not belonging to the set *GT* are either 0-arcs, or θ- arcs;
2) all 0-arcs belonging to set *GT* are properly oriented.

Then from Statement 1, it follows that vector *(x,y)* is an extreme-point solution of problem (1). Suppose $\mu_j = 0$ if $j \in GT$ and j is an opposite direction θ-arc. For the rest *j* let $\mu_j = 0$, if $y_j < b_j$ and $\mu_j = \frac{1}{a_j}$, if $y_j \geq b_j$.

For any two nodes I_1 and I_2 in the current subtree *GT* exists the only way from I_1 to I_2. The algebraic sum of variables μ_j along a path connecting the nodes I_1 and I_2, therefore, with a plus sign if the arc has a orientation coinciding with the orientation of the path, and negative otherwise, will be denote as $\left(\overrightarrow{I_1, I_2}\right)$.

Let us $\lambda_0 = 0, \quad \lambda_i = \left(\overrightarrow{0, i}\right), \quad i \in A \cup B \cup C$.

We will continue to check the following inequalities:

$$\lambda_{n_1(j)} - \lambda_{n_2(j)} + \mu_j \geq 0, \ j \in \Gamma \quad , \tag{4}$$

$$\lambda_{n_1(j)} - \lambda_{n_2(j)} \geq 0, \quad y_j > 0 \quad j \in \Gamma. \tag{5}$$

It is easy to check that conditions (4) and (5) are fulfilled for all *j* from the set *GT*.

Consider first the case when the inequality is violated for some j_0 in the conditions (4). The set of the common nodes of the paths $(0, n_1(j_0))$ and $(0, n_2(j_0))$ is not empty. Their last common node, if we move from the root of the tree, denote I_0. Then the paths $(I_0, n_1(j_0))$ and $(n_2(j_0), I_0)$ together with the arc j_0 form a cycle. This is the only cycle that is formed by joining arc j_0 to the current subtree *GT* (Fig. 4).

Next, we define some values *E* and *Y* for cycle with the set of arcs *H*. For the positive orientation of the cycle we will take the orientation of the arc j_0. If the arc *j* orientation is positive, let us put:

$$\Delta\mu_j = \mu_j, \quad \Delta y_j = b_j - y_j. \tag{6}$$

If the arc j orientation is opposite, let us put:

$\Delta\mu_j = -\mu_j$, $\Delta y_j = y_j - b_j$, if $y_j > b_j$, (7)

$\Delta\mu_j = 0$, $\Delta y_j = y_j$, if $0 < y_j \le b_j$.

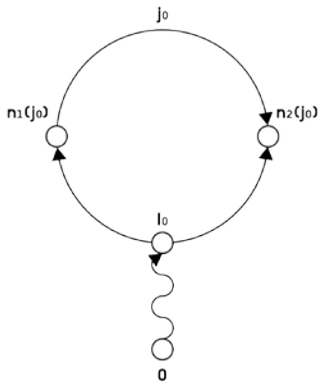

Fig. 4. The structure of the cycle that occurs when joining the arc j_0

If j is the opposite oriented 0-arc, then by putting E=0 and Y=0, we will consider them calculated. Otherwise, put them equal:

$$E = \sum_{j \in H} \Delta\mu_j, \quad Y = \min_{j \in H: \Delta_j > 0} \Delta y_j \quad (8)$$

Statement 2. If $E \ge 0$, then

$\exists j: \quad j \in (n_2(j_0), I_0)$, that arc j is θ-arc or 0-arc.

Let us prove statement 2. Let us assume the contrary. Since in path $(I_0, n_1(j_0))$ there are no 0-arcs, then from the assumptions it follows that E is the algebraic sum of variables μ_j along the whole cycle. Then

$$\lambda_{n_2(j)} - \lambda_{n_1(j)} = \left(\overrightarrow{0,n_2(j_0)}\right) - \left(\overrightarrow{0,n_1(j_0)}\right)$$
$$= \left(\overrightarrow{I_0,n_2(j_0)}\right) - \left(\overrightarrow{I_0,n_1(j_0)}\right) > \frac{1}{a_{j_0}},$$
$$E = \left(\overrightarrow{I_0,n_1(j_0)}\right) + \mu_{j_0} - \left(\overrightarrow{I_0,n_2(j_0)}\right)$$
$$\le \frac{1}{a_{j_0}} + \left(\overrightarrow{I_0,n_1(j_0)}\right) - \left(\overrightarrow{I_0,n_2(j_0)}\right) < 0.$$

It follows infidelity of made assumptions from the obtained contradiction, and the proof is complete.

Consider the case when $E \ge 0$. In this case, set of θ-arcs and 0-arcs of the path $(n_2(j_0), I_0)$ is not empty. Choose the first one.

Removing it from the subtree GT and adding the arc j_0, we get a new spanning subtree GT^*, which satisfies the conditions (4) and (5) for the plan *(x,y)*. We assume tree GT^* as the new current subtree. If we calculate a vector $(\tilde{\lambda}, \tilde{\mu})$ according to the above rules, we can prove the following statement.

Statement 3. 1) $\tilde{\lambda}_i = \lambda_i, \quad i: \quad n_2(j_0) \notin (0,i)$;

2) $\tilde{\lambda}_i = \lambda_i - \Delta, \quad i: \quad n_2(j_0) \in (0,i)$, where $\Delta = \lambda_{n_2(j)} - \lambda_{n_1(j)} - \mu_{j_0} > 0$.

Let us prove Statement 3. Item 1) is obvious, since the node i in the new subtree GT^* has the same path *(0,i)* as in subtree GT.

To justify item 2), consider the nodes of three types – x, y, z (Fig. 5).

Initially, consider the nodes of type x.

$$\tilde{\lambda}_x = \lambda_{n_1(j_0)} + \mu_{j_0} + \left(\overrightarrow{n_2(j_0),x}\right) = \lambda_{n_2(j_0)} - \lambda_{n_2(j_0)}$$
$$+ \lambda_{n_1(j_0)} + \mu_{j_0} + \left(\overrightarrow{n_2(j_0),x}\right) =$$
$$= \lambda_{n_2(j_0)} + \left(\overrightarrow{n_2(j_0),x}\right) - \Delta = \lambda_x - \Delta.$$

Next we consider the nodes of type y. Note that the path $(n_2(j_0), y)$ contains no 0-arcs and θ-arcs.

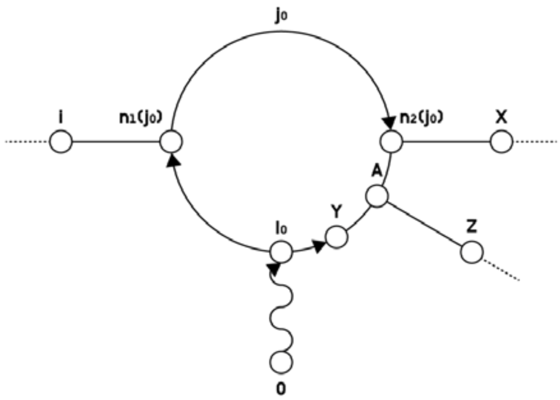

Fig. 5. The arrangement of nodes of three types – x, y, z

$$\tilde{\lambda}_y = \lambda_{n_1(j_0)} + \mu_{j_0} + \left(\overrightarrow{n_2(j_0), y}\right) =$$
$$\lambda_{n_2(j_0)} - \lambda_{n_2(j_0)} + \lambda_{n_1(j_0)} + \mu_{j_0}$$
$$+ \left(\overrightarrow{n_2(j_0), y}\right) =$$
$$= \lambda_{n_2(j_0)} + \left(\overrightarrow{n_2(j_0), y}\right) - \Delta =$$
$$\lambda_y + \left(\overrightarrow{y, n_2(j_0)}\right) + \left(\overrightarrow{n_2(j_0), y}\right) - \Delta = .$$
$$\lambda_y - \Delta$$

Next we consider the nodes of type *z*. Note that the path $(n_2(j_0), A)$ contains no 0-arcs and θ-arcs.

$$\tilde{\lambda}_z = \lambda_{n_1(j_0)} + \mu_{j_0} + \left(\overrightarrow{n_2(j_0), A}\right) + \left(\overrightarrow{A, z}\right)$$
$$= \lambda_{n_2(j_0)} - \lambda_{n_2(j_0)} + \lambda_{n_1(j_0)} + \mu_{j_0} +$$
$$+ \left(\overrightarrow{n_2(j_0), A}\right) + \left(\overrightarrow{A, z}\right) = \lambda_A + \left(\overrightarrow{A, n_2(j_0)}\right)$$
$$+ \left(\overrightarrow{n_2(j_0), A}\right) + \left(\overrightarrow{A, z}\right) - \Delta = \lambda_A + \left(\overrightarrow{A, z}\right)$$
$$- \Delta = \lambda_z - \Delta.$$

The proof is complete.

Now consider the case when $E < 0$. It is easy to see that then Y>0. Let us demonstrate how in this case to move to a new extreme-point solution of problem (1) with a smaller value of the functional. This will be carried out by cyclical change in flow according to arc j_0 orientation on the value *Y*, i.e. suppose $\tilde{y}_j = y_j + Y, \quad j \in H$, if the orientation of the arc *j* is positive and $\tilde{y}_j = y_j - Y, \quad j \in H$, if the orientation of the arc *j* is not positive and $\tilde{y}_j = y_j, \quad j \in \Gamma \setminus H$. Put also $\tilde{x}_j = x_j - \Delta\mu_j Y, \quad j \in H$ and $\tilde{x}_j = x_j, \quad j \in \Gamma \setminus H$. Values $\Delta\mu_j$ were defined above.

Statement 4. Vector $(\tilde{x}, \tilde{y})$ is a admissible solution of the problem (1).

Let us prove Statement 4. It is obvious that a cyclical change in flow does not violate the balance constraints for the flow. Therefore, we will test the feasibility of only the following three limitations:

$$\tilde{y}_j \geq 0, \quad \tilde{x}_j \geq 0, \quad \tilde{y}_j - a_j \tilde{x}_j \leq b_j .$$

Let us do it, using the rules of $\Delta\mu_j$ calculation, formulas (6) and (7) and the formulas given above to calculate the vector $(\tilde{x}, \tilde{y})$. We consider four different cases:

1) the arc *j* orientation is positive and $j \in (I_0, n_1(j_0))$ (Fig. 6).

Consider the case 1a) $y_j \geq b_j$. Then the following relationship is fair, namely:
$\tilde{y}_j = y_j + Y \geq 0,$ (1st constraint). Since
$\mu_j = \dfrac{1}{a_j}, \quad \Delta\mu_j = \mu_j = \dfrac{1}{a_j} \geq 0$, it follows the validity of the following
$\tilde{x}_j = x_j + \Delta\mu_j Y \geq x_j \geq 0,$ (2nd constraint) and
$$\tilde{y}_j - a_j \tilde{x}_j = y_j + Y - a_j(x_j + \Delta\mu_j Y) =$$
$$y_j - a_j x_j + Y(1 - a_j \Delta\mu_j) = y_j - a_j x_j \leq b_j$$
(3rd constraint).

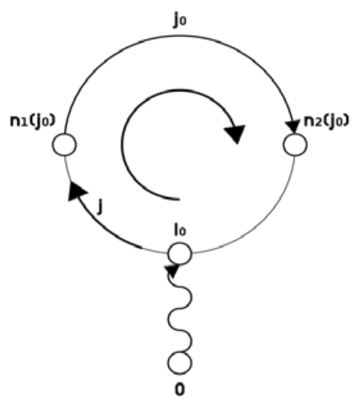

Fig. 6. The scheme of the cycle for case 1

Consider the case 1b) $y_j < b_j$. The proof of this case coincides with the proof of case 2b) below.

2) The arc *j* orientation is positive and $j \in (I_0, n_2(j_0))$ (Fig. 7).

Consider the case 2a) $y_j > b_j$. The proof of this case coincides with the proof of case 1a) above.

Consider the case 2b) $y_j \leq b_j$. Then the following relationship is fare, namely:

$\tilde{y}_j = y_j + Y \geq 0,$ (1st constraint). Since $\mu_j = 0, \quad \Delta\mu_j = \mu_j = 0, \quad \Delta y_j = b_j - y_j \geq 0$, it follows the validity of the following $\tilde{x}_j = x_j + \Delta\mu_j Y = x_j \geq 0,$ (2nd restriction), and

$\tilde{y}_j - a_j \tilde{x}_j = y_j + Y - a_j (x_j + \Delta\mu_j Y) \leq$

$y_j - a_j x_j + b_j - y_j = b_j - a_j x_j \leq b_j$

(3rd constraint).

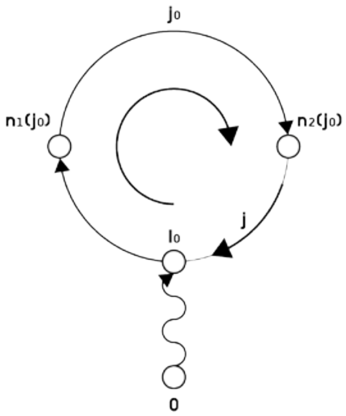

Fig. 7. The scheme of the cycle for case 2

3) Arc j orientation is not positive and $j \in (I_0, n_1(j_0))$ (Fig. 8).

Consider the case 3a) $y_j \geq b_j$. Then the following relationship is fair, namely: $\tilde{y}_j = y_j + Y \geq 0,$ (1st constraint). Since

$\mu_j = \frac{1}{a_j}, \quad \Delta\mu_j = \mu_j = \frac{1}{a_j} \geq 0$, it follows the validity of the following $\tilde{x}_j = x_j + \Delta\mu_j Y \geq x_j \geq 0,$ (2nd constraint).

$\tilde{y}_j - a_j \tilde{x}_j = y_j + Y - a_j (x_j + \Delta\mu_j Y) =$

$y_j - a_j x_j + Y(1 - a_j \Delta\mu_j) = y_j - a_j x_j \leq b_j$

(3rd constraint).

Consider the case 3a) $y_j > b_j$. Then the following relationship is fair, namely: $\Delta y_j = y_j - b_j > 0$,

$\tilde{y}_j = y_j - Y \geq y_j - \Delta y_j = b_j \geq 0,$

(1st constraint). Since

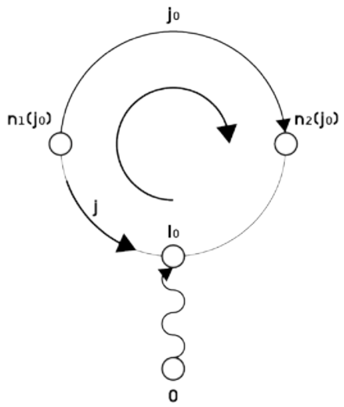

Fig. 8. The scheme cycle for case 3

$\mu_j = \frac{1}{a_j}, \quad \Delta\mu_j = -\mu_j = -\frac{1}{a_j} \geq 0$, it follows the validity of the following ratio

$\tilde{x}_j = x_j + \Delta\mu_j Y = x_j - \frac{1}{a_j} Y \geq$

$\frac{1}{a_j}(a_j x_j - \Delta y_j) = \frac{1}{a_j}(a_j x_j - y_j + b_j) \geq 0,$

(2nd constraint)),

$\tilde{y}_j - a_j \tilde{x}_j = y_j - Y - a_j (x_j + \Delta\mu_j Y) =$

$y_j - a_j x_j - Y(1 + a_j \Delta\mu_j) = y_j - a_j x_j \leq b_j$

(3rd constraint).

Consider the case of 3b) $y_j \leq b_j$. In this case the following relationship is fair, namely:

$\Delta y_j = y_j \geq 0,$

$\tilde{y}_j = y_j - Y \geq y_j - \Delta y_j = 0,$

(1st constraint). Since

$\mu_j = 0, \quad \Delta\mu_j = -\mu_j = 0$, it follows the validity of the following

$\tilde{x}_j = x_j + \Delta\mu_j Y = x_j \geq 0,$ (2nd constraint), and

$\tilde{y}_j - a_j \tilde{x}_j = y_j - a_j x_j - Y \leq b_j - a_j x_j \leq b_j$

(3rd constraint).

4) arc j orientation is not positive and $j \in (I_0, n_2(j_0))$ (Fig. 9).

Consider the case of 4a) $y_j > b_j$.

The proof of this case coincides with the proof of case 3a), given above.

Consider the case of 4b) $y_j \le b_j$.

The proof of this case coincides with the proof of case 3b) above.

The proof is complete.

Statement 5. $\sum_{j \in \Gamma} x_j > \sum_{j \in \Gamma} \tilde{x}_j$.

Let us prove Statement 5.

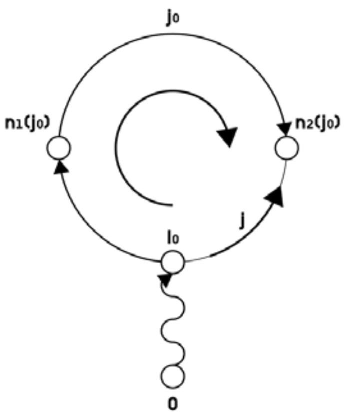

Fig. 9. The scheme cycle for case 4

$$\sum_{j \in \Gamma} \tilde{x}_j - \sum_{j \in \Gamma} x_j = \sum_{j \in \Gamma} (\tilde{x}_j - x_j) = \sum_{j \in H} (\tilde{x}_j - x_j) = Y \sum_{j \in H} \Delta\mu_j = YE < 0.$$

The Statement 5 is proved.

Statement 6. There is an arc *j*, $j \in H$, such that it is either an θ-arc, or an 0-arc.

Let us prove Statement 6. We choose an arbitrary index *j*, such that $j \in Arg \min_{j \in H : \Delta y_j > 0} \Delta y_j$. Then $0 < \Delta y_j = Y$. Let *j* has positive orientation, then we have the following: $\tilde{y}_j = y_j + Y = y_j + \Delta y_j = y_j + (b_j - y_j) = b_j$

That is, this arc is θ-arc. Suppose now that *j* has the opposite orientation. Then, if $y_j > b_j$, then

$\tilde{y}_j = y_j - Y = y_j - \Delta y_j = y_j - (-b_j + y_j) = b_j$

That is, this arc is an θ-arc. If $0 < y_j \le b_j$, then $\tilde{y}_j = y_j - Y = y_j - \Delta y_j = y_j - y_j = 0$.

That is, this arc is a 0-arc. The Statement 6 is proved.

Now we show how to choose an arc j_1 is to be deleted from the set *GT*. The path $(n_1(j_0), I_0)$ can contain only collinear with it 0-arcs. If their set is not empty, then we choose j_1 as the last of them.

Otherwise as j_1 choose any of the θ-arcs, which are in force Statement 6 in this case is not empty. Removing the arc j_1 from the set *GT*, and adding an arc j_0, we get a new spanning subtree *GT**, which for plan $(\tilde{x}, \tilde{y})$ satisfies the conditions 1) and 2). Note that because of conditions (4) and the Statement 1 the plan is an extreme-point solution of problem (1).

Now consider the case when inequality j_0 of conditions (5) is violated. Consider the same loop as above in the opposite direction of the arc j_0. Calculate the values *E* and *Y* according to the formulas (6), (7) and (8). In this case following statement is fare.

Statement 7. If $E \ge 0$, then there exists an arc *j* that $j \in (n_1(j_0), I_0)$ and *j* is an θ-arc or 0-arc.

The proof of this Statement is similar to that of Statement 2.

If $E \ge 0$ let us delete from the tree GT the first θ-arc or 0-arc, belonging to the path $(n_1(j_0), I_0)$ and add arc j_0. We get a new spanning subtree *GT**, which satisfies conditions 1) and 2). For the corresponding vector $(\tilde{\lambda}, \tilde{\mu})$ the following statement is true.

Statement 8.

1) $\tilde{\lambda}_i = \lambda_i, \quad i: \quad n_1(j_0) \notin (0, i)$;

2) $\tilde{\lambda}_i = \lambda_i - \Delta, \quad i: \quad n_1(j_0) \in (0, i)$, where $\Delta = \lambda_{n_2(j)} - \lambda_{n_1(j)} > 0$.

The proof of this Statement is similar to Statement 3.

In the case when *E<0*, let us do cyclic change of the flow in the opposite direction of the arc j_0 on the value of *Y*. We obtain a new admissible solution $(\tilde{x}, \tilde{y})$ of the problem (1) by the same formulas as above. For the solution $(\tilde{x}, \tilde{y})$ Statements 4 and 5 will be fair.

Now we show how to choose an arc j_1 to be deleted from *GT*. The path $(n_2(j_0), I_0)$ can contain only collinear with it 0-arcs. If their set is not empty, then we choose the last one. Otherwise consider the arc j_0. If it is a 0-arc, then leave the current subtree without changes. Otherwise, consider the path $(I_0, n_1(j_0))$ that may contain only collinear with it 0-arcs. If their set is not empty, then we choose j_1 as the last of them. Otherwise as j_1 choose any of the θ-arcs of the cycle, which in this case, as to Statement 6 is not empty. Removing the arc j_1 from the set *GT*, and adding an arc j_0, we get a new spanning subtree *GT**, which for solution $(\tilde{x}, \tilde{y})$ satisfies the conditions 1) and 2). So $(\tilde{x}, \tilde{y})$ is an extreme-point solution of problem (1).

Thus, at each step there is a transition either to a new extreme-point solution with a smaller value of the functional or to a new spanning subtree, the values of the components of the vector λ which is no more than, and at least for one node strictly smaller, than the vector components λ of the previous spanning tree. This eliminates the possibility of looping by the above algorithm.

Due to the fact that there is a finite number of different spanning subtrees and extreme-point solutions for problem (1), we get an extreme-point solution *(x,y)* and a spanning subtree *GT* that satisfy the conditions 1) and 2), such that the associated vector (λ, μ) satisfies the inequalities (4) and (5) in a finite number of steps.

Consider the following vector $(\lambda, \tilde{\mu})$, such that $\tilde{\mu}_j = \lambda_{n_2(j)} - \lambda_{n_1(j)}$, if $j \in \Gamma \setminus \Gamma T$, $j-\theta$-arc and $\tilde{\mu}_j = \mu_j$ for the other *j*. Note that due to relations (4) and (5) $0 \le \mu_j \le \frac{1}{a_j}, \quad j \in \Gamma$.

It is easy to check that the vector $(\lambda, \tilde{\mu})$ is an admissible solution of the problem (2) and associated by conditions (3) with extreme-point solution *(x,y)*. By theorem 1 it follows that vector *(x,y)* is the optimal solution of the problem (1).

It remains to show the existence of some initial feasible solution *(x,y)* and a spanning subtree *GT* for which the conditions 1) and 2) are fulfilled. It is easy to do, relying on the following statement.

Statement 9. There is a connecting subtree of the original graph with the proper orientation (Definition 3) of all its arcs.

Let us prove statement 9. For each node of the initial graph an oriented path exists from root to this node, otherwise the node could be initially excluded from consideration, together with the incident arcs. We will take the root of the tree as the initial subtree. Connect the root by oriented way with an arbitrary node, not belonging to the subtree. Consider the plot of this path from this node to the first node belonging to the subtree. A new subtree contains at least one node more than the previous one, and all its arcs are properly oriented. Continuing the above procedure, for a finite number of steps we will construct a spanning subtree with the proper orientation of all its arcs. The proof is complete.

Problem (1) was considered by us in the assumption that $a_j > 0, \quad j \in \Gamma$. Let us now consider the problem in the general case, when $a_j \ge 0, \quad j \in \Gamma$. Note also that the case $a_j < 0$ has no meaningful sense from the point of view of the consideration of the *x* variables as resource variables. In the general case the problem has the following form:

$$\min_{x,y}\left(\sum_{j \in \Gamma_1} x_j\right), \tag{9}$$

$$\sum_{j \in C(i)} y_j - \sum_{j \in D(i)} y_j = 0, \quad i \in B,$$

$$\sum_{j \in D(i)} y_j - \sum_{j \in C(i)} y_j = d_i, \quad i \in C,$$

$$y_j \le b_j, \quad j \in \Gamma_2,$$

$$y_j - a_j x_j \le b_j, \quad j \in \Gamma_1,$$

$$x_j \ge 0, \quad j \in \Gamma_1, \quad y_j \ge 0, \quad j \in \Gamma.$$

Where

$$\Gamma_1 = \left\{ j \in \Gamma : \quad a_j > 0 \right\},$$

$$\Gamma_2 = \left\{ j \in \Gamma : \quad a_j = 0 \right\}.$$

Problem (9), generally speaking, may not have admissible solutions. Now let us consider an auxiliary problem by introducing additional variables $x_j \ge 0 \quad j \in \Gamma_2$ and capacity functions $\delta x_j + b_j \ge 0 \quad j \in \Gamma_2$,

where

$$\delta = \left(2 \sum_{j \in \Gamma_1} \frac{1}{a_j}\right)^{-1}.$$

The mathematical formulation of the auxiliary problem has the following form:

$$\min_{x,y}(\sum_{j\in\Gamma} x_j), \qquad (10)$$

$$\sum_{j\in C(i)} y_j - \sum_{j\in D(i)} y_j = 0, \quad i\in B,$$
$$\sum_{j\in D(i)} y_j - \sum_{j\in C(i)} y_j = d_i, \quad i\in C,$$
$$y_j \le \delta x_j + b_j, \quad j\in\Gamma_2,$$
$$y_j - a_j x_j \le b_j, \quad j\in\Gamma_1,$$
$$x_j \ge 0, \quad y_j \ge 0, \quad j\in\Gamma.$$

Problem (10) refers to the type of problems which can be solved by algorithm discussed in detail above. Therefore, it can be solved by method of generalized potentials.

The following theorem is fair.

Theorem 2. Let *(x*,y*)* is optimal solution of problem (10). If it exists $i\in\Gamma_2$, such that $x_i^* > 0$, the set of feasible solutions of problem (9) is empty. Otherwise, vector *(x*,y*)* is optimal solution of problem (9).

Prove theorem 2. Let *(x*,y*)* is optimal solution of problem (10). Suppose that the set $B=\{i\in\Gamma_2: \quad x_i^*>0\}$ is not empty. If the solution *(x*,y*)* is not an extreme-point of the feasible set, it can be represented as some convex combination of optimal extreme-points. Then it can be stated that if $j_0\in B$, then an optimal extreme-point exists for which $x_{j_0}^* > 0$. That is, a set *B* that is not empty. Thus, we can assume without loss of generality that *(x*,y*)* is the optimal extreme-point.

Now assume that the set of feasible solutions is not empty and the vector *(x,y)* is a feasible solution of the problem (9). If $A=\sum_{i\in\Gamma_1} x_i$ it is easy to show that problem (9) is equivalent to the problem on the set:

$$0\le y_j \le b_j, \quad j\in\Gamma_2, \quad 0\le y_j \le Aa_j + b_j,$$

$$j\in\Gamma_2, \quad 0\le x_j \le A, \quad j\in\Gamma_1$$

Consequently, the problem has a solution because of the Weierstrass theorem. Since the problem (9) is a linear programming problem, then it has an optimal extreme-point, which we denote as $(\bar{x},\bar{y})$. If now we complement the vector $(\bar{x},\bar{y})$ by components $x_j=0, \quad j\in\Gamma_2$, the resulting vector $(\tilde{x},\tilde{y})$ is an extreme-point of problem (10). This follows from the admissibility of the vector $(\tilde{x},\tilde{y})$ and the fact that the system of columns corresponding to positive components is not changed, that is still linearly independent.

From the theory of linear programming, it is known, that a sequence of extreme-points exists, starting with arbitrary initial point $(\tilde{x},\tilde{y})$ and ending with the optimal solution *(x*,y*)*. Adjacent members of this sequence are adjacent extreme-points (that is, their bases are different by a single vector), and the value of the functional at these points does not increase monotonically.

Directly from the above algorithm of the generalized potentials description it follows that the transition from extreme-point to a neighboring extreme-point with not bigger value of the functional is proceeded by varying vectors y and x along some cycle for which $E\le 0$ and $Y>0$.

Since for the vector $(\tilde{x},\tilde{y})$ corresponding set *B* is empty, and for the vector *(x*,y*)* is not empty, then in the sequence a pair of neighboring points (x_1,y_1) and (x_2,y_2), exists for which the set *B* is empty and not empty, respectively.

This transition can be carried out only for the loop containing the properly oriented arc j_0 of the set G_2. But then $\Delta\mu_{j_0} = \frac{1}{\delta} = 2\sum_{j\in\Gamma_1}\frac{1}{a_j}$. If the cycle contains arc *j* of the set G_2 with the opposite orientation, then $\Delta\mu_j = 0$. Let *r* arcs in a cycle of the set G_2 have the proper orientation. Then the following is fair:

$$E=\sum_{j\in H}\Delta\mu_j \ge r\frac{1}{\delta} - \sum_{j\in\Gamma_1}\frac{1}{a_j} > 0.$$

Therefore, such a transition is impossible. And, therefore, the assumption of non-emptiness of the set of feasible solutions of problem (9) is incorrect. Thus, the first part of the theorem is proved.

The justice of the second part of the theorem follows directly from the fact that the vector *(x*,y*)* is an admissible solution of the problem (9), and the problem (9) is a restriction of problem (10) for a subset of the set of the admissible solutions. The theorem is proved.

4 The problem with undefined factors

A finite number of undefined factors of a scenario type that affect the form of the arc capacity functions and production capacities are considered. The mathematical formulation of the problem of optimal synthesis with uncertain factors is the following:

$$\min_{x,y}\left(\sum_{j\in\Gamma} x_j\right), \qquad (11)$$

$$\sum_{j\in C(i)} y_j^k - \sum_{j\in D(i)} y_j^k = 0, \quad i\in A\cup B,$$

$$\sum_{j\in D(i)} y_j^k - \sum_{j\in C(i)} y_j^k \ge d_i, \quad i\in C,$$

$$y_j^k - a_j^k x_j \le b_j^k, \quad j\in\Gamma,$$

$$x_j \ge 0,\ y_j^k \ge 0,\ j\in\Gamma,\ k=1,\dots,l.$$

The problem (11) is a linear programming problem. The dual problem to it has the following form:

$$\max_{\lambda,\mu}\left(\sum_{i\in C} d_i\left(\sum_{k=1}^{l}\lambda_i^k\right) - \sum_{j\in\Gamma}\sum_{k=1}^{l}\mu_j^k b_j^k\right) \qquad (12)$$

$$1-\sum_{k=1}^{l}\mu_j^k a_j^k \ge 0, \quad j\in\Gamma, \qquad (13)$$

$$\lambda_{n_2(j)} - \lambda_{n_1(j)} - \mu_j^k \le 0, \quad j\in\Gamma\setminus C(0),$$

$$\lambda_{n_2(j)} - \mu_j^k \le 0, \quad j\in C(0)$$

$$\mu_j^k \ge 0,\ j\in\Gamma,\ \lambda_i^k \ge 0,\ i\in C,\ k=1,\dots,l.$$

Taking into account the restrictions (13) and the fact that $\mu_j^k \ge 0$, we can add restrictions

$$\mu_j^k a_j^k \le 1, \quad j\in\Gamma, \quad k=1,\dots,l$$

to the constraints of the problem (12). The problem (12) remains unchanged. If we introduce auxiliary variables $z, z^1,\dots,z^l$ to reduce the constraints to the system of equations, then the constraint matrix of problem (12) takes the form shown in Fig. 10.

λ^1	μ^1	z^1	λ^2	μ^2	z^2	…	z		b
0	A^1	0	0	A^2	0		E	x	1 … 1
IN^T	E	-E	0	0	0		0	y^1	0 … 0
…	…	…	…	…	…	…	…	…	…
0	0	0	0	IN^T	E	-E	0	y^l	0 … 0

Fig. 10. Structure of the constraint matrix.

Where E is the unit matrix, IN is the matrix of incidence of the graph of the network, and the matrix A^i is a diagonal matrix of the coefficients a of the following form (Fig. 11):

$$A^i = \begin{pmatrix} a_1^i & 0 & 0 & 0 \\ 0 & . & 0 & 0 \\ 0 & & . & 0 \\ 0 & 0 & 0 & a_n^i \end{pmatrix}$$

Fig. 11. Structure of matrix A^i.

The constraint matrix of problem (12) is a block matrix with a group of connecting rows. For such problems, the application of decomposition methods is effective. Consider the well-known Danzig-Wolfe decomposition method [10]. We will not describe here the general scheme of the method. We note only that its essence consists in replacing the solution of the original problem by solving a series of problems of lower dimension [11]. In our case, at each iteration it is necessary to solve the problem of the following form:

$$\max_{\lambda,\mu}\left(\sum_{i\in C} d_i\lambda_i^k - \sum_{j\in\Gamma}\mu_j^k\tilde{b}_j^k\right) \qquad (14)$$

$$1-\mu_j^k a_j^k \ge 0, \quad j\in\Gamma,$$

$$\lambda_{n_2(j)} - \lambda_{n_1(j)} - \mu_j^k \le 0, \quad j\in\Gamma\setminus C(0),$$

$$\lambda_{n_2(j)} - \mu_j^k \le 0, \quad j\in C(0),$$

$\mu_j^k \geq 0,\ j \in \Gamma,\ \lambda_i^k \geq 0,\ i \in C,\ k = 1,...,l.$

Where are coefficients $\tilde{b}_j^k$ that vary by iteration. The problem dual to the problem (14) has the following form:

$$\min_{\tilde{x}, y}(\sum_{j \in \Gamma} x_j), \qquad (15)$$

$$\sum_{j \in C(i)} y_j^k - \sum_{j \in D(i)} y_j^k = 0, \quad i \in A \cup B,$$

$$\sum_{j \in D(i)} y_j^k - \sum_{j \in C(i)} y_j^k \geq d_i, \quad i \in C,$$

$$y_j^k - a_j^k x_j \leq \tilde{b}_j^k, \quad j \in \Gamma,$$

$$x_j \geq 0,\ y_j \geq 0,\ j \in \Gamma,\ k = 1,...,l.$$

Let's consider the sets $\Gamma_1 = \{ j \in \Gamma:\ \tilde{b}_j^k \geq 0 \}$ and $\Gamma_2 = \Gamma \setminus \Gamma_1$. For $j \in \Gamma_2, x_j \geq -\frac{\tilde{b}_j^k}{a_j^k} = x_j^0 \geq 0.$

Let's make the change of variables:

$x_j = \tilde{x}_j + x_j^0,\ j \in \Gamma_2, x_j = \tilde{x}_j,\ j \in \Gamma_1.$

The problem (15) in new variables looks as:

$$\min_{\tilde{x}, y}(\sum_{j \in \Gamma} \tilde{x}_j), \qquad (16)$$

$$\sum_{j \in C(i)} y_j^k - \sum_{j \in D(i)} y_j^k = 0, \quad i \in A \cup B,$$

$$\sum_{j \in D(i)} y_j^k - \sum_{j \in C(i)} y_j^k \geq d_i, \quad i \in C,$$

$$y_j^k - a_j^k x_j \leq \tilde{b}_j^k, \quad j \in \Gamma_1,$$

$$y_j^k - a_j^k \tilde{x}_j \leq 0, \quad j \in \Gamma_2,$$

$$\tilde{x}_j \geq 0,\ y_j \geq 0,\ j \in \Gamma,\ k = 1,...,l.$$

The problem (16) admits a solution by the method of generalized potentials expounded earlier. In addition, along with the optimal solution $(\tilde{x}, y)$ in the course of implementing the method, we simultaneously obtain a vector $(\lambda, \tilde{\mu})$ that is a solution of the problem dual to problem (16). We set

$$\mu_j^k = \tilde{\mu}_j^k,\ j \in \Gamma\ ,\ \mu_j^k = \frac{1}{a_j^k},\ j \in \Gamma_2.$$

Statement 10. The vector (λ, μ) is the optimal solution of the problem (14).

Let us prove statement 10. It is not difficult to see that the vector (λ, μ) is an admissible solution of the problem (14). Let the vector *(x, y)* be a solution of the problem (11), and let the vector $(\tilde{x}, y)$ be a solution of the problem (16). We show that the vectors (x, y) and (λ, μ) are related by (13). This proves the assertion to be proved.

1) Let $x_j > 0$ and $j \in \Gamma_1$, then

$$\mu_j^k = \tilde{\mu}_j^k, \quad \tilde{\mu}_j^k a_j^k = 1$$

and hence, $\mu_j^k a_j^k = 1$. If $j \in \Gamma_2$, then

$$\mu_j^k a_j^k = 1.$$

2) Let $y_j > 0$. If $\mu_j^k = \tilde{\mu}_j^k$, then $\lambda_{n_2(j)} - \lambda_{n_1(j)} - \tilde{\mu}_j^k = 0$ and, hence, $\lambda_{n_2(j)} - \lambda_{n_1(j)} - \mu_j^k = 0$. If $\mu_j^k \neq \tilde{\mu}_j^k$, then $x_j = 0$ and, hence, $y_j = 0$.

3) Let $y_j^k - a_j^k x_j < b_j^k$, then $j \in \Gamma_1$ and, hence, $\mu_j^k = \tilde{\mu}_j^k = 0$. Statement is proved.

Thus, applying the Danzig-Wolfe decomposition method and the algorithm described above, one can obtain a solution (λ, μ) of the problem (12).

Let us dwell on the question of how to use the obtained vector (λ, μ) to find the vector *(x, y)*, which is the optimal solution of the problem (11).

Consider the following problem, which is a modification of the problem (11):

$$\min_{x,y}(\sum_{j\in\Gamma} x_j), \qquad (17)$$

$$\sum_{j\in C(i)} y_j^k - \sum_{j\in D(i)} y_j^k \le 0, \quad i\in A\cup B,$$

$$\sum_{j\in D(i)} y_j^k - \sum_{j\in C(i)} y_j^k \ge d_i, \quad i\in C,$$

$$y_j^k - a_j^k x_j \le b_j^k, \quad j\in\Gamma,$$

$$x_j \ge 0,\ y_j \ge 0,\ j\in\Gamma,\ k=1,...,l.$$

The following statement is true.

Statement 11.

1) The optimal values of the functions being minimized in problems (11) and (17) coincide.

2) If the vector $(x,\tilde{y})$ is a solution of the problem (17), then there exists a vector (x, y), which is a solution of the problem (11).

Let us prove statement 11. We begin with the proof of 2). Suppose that for some vertex i the restriction is satisfied as a strict inequality, that is.

$$\sum_{j\in D(i)} y_j^k - \sum_{j\in C(i)} y_j^k = p > 0.$$

Then there exists some path consisting of arcs with nonzero flows from the vertex 0 to the vertex i. Let Δy there be a minimal flow flowing along the arcs of the given path. Then we set Y = min (Δy, p). We reduce the values of the streams of arcs of the given path by the amount Y. In this case, all constraints of the problem (17) remain satisfied. And the restrictions of equality of a species will remain $\sum_{j\in D(i)} y_j^k - \sum_{j\in C(i)} y_j^k = 0$.

If p = 0, then, for the node i, we have the equality of the incoming and outgoing streams. Otherwise, repeat the procedure for selecting the path. It is clear that applying the described algorithm in a finite number of steps will result in equalization of the incoming and outgoing streams for the i-th, and, consequently, for any other node. Thus, in a finite number of steps, a solution (x, y) satisfying the constraints of the problem (11) will be constructed.

We now prove part 1). Let A be the value of the minimum in problem (11), and B the value of the minimum in problem (17). We note that the values of the functionals for the vectors $(x,\tilde{y})$ and (x, y) are the same. From this and from 2) it follows that. $A\le B$.Since the set of admissible solutions of the problem (17) is wider than of the problem (11), then $A\ge B$. Whence finally we have A = B. The statement is proved.

Since the transition from solution $(x,\tilde{y})$ to solution (x, y) is very simple, we will only seek solutions of problem (16). We write the problem dual to the problem (16) in the canonical form:

$$\max_{\lambda,\mu}(\sum_{i\in C} d_i(\sum_{k=1}^{l}\lambda_i^k) - \sum_{j\in\Gamma}\sum_{k=1}^{l}\mu_j^k b_j^k) \qquad (18)$$

$$\sum_{k=1}^{l}\mu_j^k a_j^k + z_j = 1, \quad j\in\Gamma,$$

$$\lambda_{n_2(j)} - \lambda_{n_1(j)} - \mu_j^k + v_j^k = 0, \quad j\in\Gamma\setminus C(0)$$

$$\lambda_{n_2(j)} - \mu_j^k + v_j^k = 0, \quad j\in C(0),$$

$$\mu_j^k \ge 0,\ j\in\Gamma,\ \lambda_i^k \ge 0,\ z_j^k \ge 0,\ v_j^k \ge 0,$$

$$i\in A\cup B\cup C,\ k=1,...,l.$$

This problem differs from the problem (12) by the non-negativity of all its variables. The solution (λ,μ) of problem (12) in view of its nonnegativity is simultaneously a solution of problem (18). From the vector (λ,μ) found, the components of the vectors z and v are uniquely determined. Thus, we finally have a vector (λ,μ,z,v) that is a solution of the problem (18). However, the solution (λ,μ,z,v) can, firstly, not be the extreme point of a feasible set, and secondly, it is a degenerate extreme point.

In the first case, we single out the maximal linearly independent system among the columns corresponding to the positive components of the vector (λ,μ,z,v). We supplement it to a basis B. It is known that there exists an optimal extreme point with basis B [3].

In the second case, the rank of the system of columns corresponding to positive components is less than the rank of the matrix of the constraint system. In this case it is necessary to supplement it to a complete basis B.

In both cases, the formation of the basis B is equivalent in complexity to reducing the matrix to a triangular form.

Let b be the vector of the right-hand sides of the constraints of the problem (17). From the theory of duality of linear programming, it is known that if there is a pair of dual-purpose problems

$$\max_{\lambda}(b,\lambda), \qquad \min_{x}(c,x),$$
$$A\lambda = c\,, \qquad A^T x \ge b\,.$$
$$\lambda \ge 0\,.$$

and B is the basis of the optimal solution of one of the problems λ^*, then the vector $x^* = (B^T)^{-1} b_B$ is the optimal solution of the other problem. Using this statement in our case, we can assert that the vector $(x,\tilde{y}) = (B^T)^{-1} b_B$ is the optimal solution of the problem (17).

5 Conclusion

Thus it has been developed and validated a new efficient algorithm for linear separable problem of synthesis of communication networks, titled "method of generalized potentials", which is close to the well-known method of potentials for solving the classical transportation problem [3].

The algorithm was developed for the Gale problem on demand and supply with the arcs capacities, depending on the distribution of resources. It is shown that from a mathematical point of view the task from multiple points of production and limited stocks of product can be reduced to the task with a single item of production with an unlimited supply of the product. The algorithm of the method of generalized potentials applied to the reduced problem. The developed algorithm of transition to the solution of the original problem, proved the finiteness of the algorithm and the inability of the looping situations.

An original algorithm is constructed on the basis of the Danzig-Wolfe decomposition method and the method of generalized potentials for large-dimensional problems with indeterminate factors, which allows one to reduce the solution of the original large-dimensional problem to solving a series of problems that are considerably simpler from a computational point of view.

An algorithm for the synthesis of the optimal solution of the original problem is proposed. The proposed algorithm makes it possible to substantially increase the dimensionality of the problems being solved.

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